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The wave resistance of a two-dimensional body moving forward in a two-layer fluid

O.V. MOTYGIN and N.G. KUZNETSOV
Laboratory on Mathematical Modelling in Mechanics, Institute of Problems of Mechanical Engineering, Russian Academy of Sciences, V.O., Bol'shoy pr. 61, St. Petersburg, 199178, Russia

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Abstract. A two-dimensional body moves forward with constant velocity in an inviscid, incompressible fluid under gravity. The fluid consists of two layers having different densities, and the body is totally submerged in one of them. The resulting fluid motion is assumed to be steady state in a coordinate system attached to the body. The boundary-value problem for the velocity potential is considered in the framework of linearized water-wave theory. The asymptotics of the solution at infinity is obtained with the help of an integral representation, based on the explicitly known Green function. The theorem of unique solvability is formulated, and the method applied to prove it is briefly explained (the detailed proof is given in another work). An explicit formula for the wave resistance is derived and discussed. A numerical example for the wave resistance serves to illustrate the so-called "dead-water" phenomenon.

Key words: wave resistance, dead water, Green's function, far-field asymptotics

1. Introduction

The present paper is concerned with linear waves resulting from the forward motion of a cylinder in an inviscid incompressible stratified fluid under gravity. Two superposed fluids of different density occupy horizontal layers, and one of them contains the body. The upper layer is bounded from above by the free surface and has a finite depth, while the depth of the lower layer is infinite. The aim of this work is to derive an explicit formula for the wave-making resistance. The latter includes a contribution from the internal waves which arise on the interface for some range of the parameters involved. The integral-equation technique for finding solutions to the corresponding boundary-value problems is developed in a separate paper [1], where proofs of the existence and uniqueness theorems are given. Here we restrict ourselves to the formulation of these results and a brief comment.

Forward motion in a two-layer fluid is connected with the so-called "dead-water" phenomenon. It manifests itself in abnormal resistance occasionally experienced by ships in Norwegian fjords and was described almost one hundred years ago [2, pp. 1–152]. First explanations in terms of moving surface pressures (distributed or concentrated) appeared soon after that [3, 4]. In the later 1950s the first papers [5, 6, 7] had appeared that treated steady-state wave motion due to sources in a two-layer fluid. By that time considerable advances had been made in the theory of linear waves due to a body moving forward in a homogeneous fluid [8]. Since then some articles have been published which consider surface and internal waves simultaneously (see a survey [9]). Nevertheless, no general theory has been developed in this area despite the growing interest caused by applications (see *e.g.* [10, 11, 12]). The greater part of published papers is concerned with the numerical analysis of hydrodynamic forces, and in particular, of wave resistance (see [11, 12, 13] and references cited therein). The existence of solutions

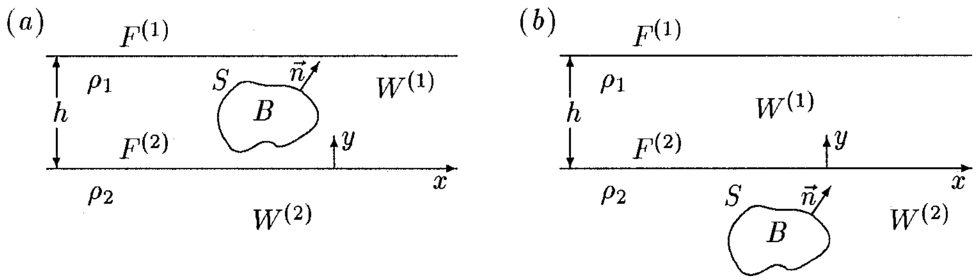

Figure 1. Definition sketch of the coordinate system and geometric notations when the body moves in the upper (a) or lower (b) layer.

is shown in [14] and [15] only for bodies close in some sense to a circular cylinder placed beneath or above the interface. The lack of rigorous mathematical treatment of the problem stimulated the present investigation of the far-field asymptotics and wave resistance. It should be added that the above-mentioned paper [1] gives almost exhaustive results on the existence and uniqueness for the case where a cylinder of arbitrary cross-section is totally immersed in one of the layers.

The plan of the paper is as follows. The governing equations are presented in Section 2 along with the formulation of the existence and uniqueness theorems. In Section 3 the general form of the free-stream velocity potential is obtained, and for the velocity potential describing the flow about an arbitrary cylinder immersed in one of the layers the integral representation and the asymptotics at infinity are derived. Two formulae for the wave resistance are deduced in Section 4. The first of these expresses the resistance in terms of the asymptotic coefficients which are proportional to the amplitudes of surface and internal waves far downstream. The second formula connects the wave resistance with energy losses. In Section 4 we also give a numerical example. The Appendix contains the necessary information on Green's functions. The corrected representations with regularized integral terms (regularization of integrals involved in Green's functions was not done in [14] and [15]) are given for all possible positions of the point source and the observation point. Furthermore, the asymptotic formulae at infinity are presented along with some hints on the derivation method.

2. Statement, solvability and uniqueness of the problem

Consider a cylinder in a steady-state forward motion in a direction orthogonal to its generators. The body moves beneath the free surface of an inviscid incompressible heavy fluid consisting of two superposed horizontal layers having different densities. The upper layer of finite depth h has a density ρ_1. The lower fluid of density $\rho_2 > \rho_1$ is unbounded from below. The body is totally immersed in the upper (see Figure 1a) or in the lower (see Figure 1b) layer.

The origin is placed in the unperturbed interface. Let the cross-section B of the cylinder be a simply connected domain bounded by a closed $C^{2,\alpha}$-curve $S (0 < \alpha < 1)$. By $L^{(1)} = \{(x, y) : 0 < y < h\}$ and $L^{(2)} = \{(x, y) : y < 0\}$ we denote the upper and lower layers, respectively, in the absence of the body. Let $W^{(1)} = L^{(1)} \setminus \overline{B}$ and $W^{(2)} = L^{(2)} \setminus \overline{B}$ denote domains occupied by the upper and lower fluid, respectively; at rest the free surface of the upper fluid is $F^{(1)} = \{(x, y) : y = h\}$, and $F^{(2)} = \{(x, y) : y = 0\}$ is the interface between the fluids. We put $\varepsilon = \rho_2 \rho_1^{-1} - 1$.

The usual linear model describing waves (referred to as the two-layer Neumann–Kelvin problem) is as follows. Assuming that $B \subset L^{(i)}$, find functions $u^{(i)} \in C^{2,\alpha}(\overline{W^{(i)}}), i = 1, 2,$ satisfying the boundary-value problem:

$$\nabla^2 u^{(i)} = 0 \quad \text{in} \quad W^{(i)}, \quad i = 1, 2, \tag{2.1}$$

$$u^{(1)}_{xx} + \nu u^{(1)}_y = 0 \quad \text{on} \quad F^{(1)}, \tag{2.2}$$

$$\rho_1 [u^{(1)}_{xx} + \nu u^{(1)}_y] = \rho_2 [u^{(2)}_{xx} + \nu u^{(2)}_y] \quad \text{on} \quad F^{(2)}, \tag{2.3}$$

$$u^{(1)}_y = u^{(2)}_y \quad \text{on} \quad F^{(2)}, \tag{2.4}$$

$$\partial u^{(i)} / \partial n = f \quad \text{on} \quad S, \tag{2.5}$$

$$\sup_{W^{(i)}} |\nabla u^{(i)}| < \infty, \quad i = 1, 2, \tag{2.6}$$

$$\lim_{x \to +\infty} |\nabla u^{(i)}| = 0, \quad i = 1, 2. \tag{2.7}$$

Here ***n*** is the unit normal to S directed into the fluid, $\nu = gU^{-2}$ is a positive parameter, where g is the acceleration due to gravity and U is the body's constant speed. When B is a rigid body, then $f = U \cos(n, x)$ in (2.5). The condition (2.7) means that the body moves in the direction of the x-axis. The function $u^{(1)}$ $(u^{(2)})$ is the velocity potential of the steady state motion of the upper (lower) fluid in the frame of reference attached to the body. Clearly, the functions $u^{(1)}$ and $u^{(2)}$ are defined up to constant (generally different) terms.

In the limit as $\varepsilon \to 0$ the fluid becomes homogeneous $(\rho_1 = \rho_2)$. Then (2.3) and (2.4) imply that $u^{(1)}$ and $u^{(2)}$ are restrictions of the velocity potential given on the whole fluid domain $W^{(1)} \cup F^{(2)} \cup W^{(2)}$, to the domains $W^{(1)}$ and $W^{(2)}$, respectively. This allows us to verify the results obtained for (2.1)–(2.7) as follows. We allow ε to tend to zero and check whether these results are the same as those for the forward motion of a body in a homogeneous fluid (see [8, Section 26, p. 169], [16, Section 249, p. 415], [17, 18]) which are known.

It is well-known (see [9, 13] and the Appendix) that the source potential changes its form at the critical value $\nu_* = (1 + \varepsilon)/\varepsilon h$ of the parameter ν (see Figure 2). This means that the flow due to a source changes its structure at $\nu = \nu_*$. For $\nu > \nu_*$ (the region D_2 on Figure 2) the internal waves exist along with the waves on the free surface of the upper fluid, but for $\nu < \nu_*$ (the region D_1) there are no internal waves. In the next section we prove rigorously that the flow about an arbitrary cylinder has the same behaviour.

The problem (2.1)–(2.7) depends on two positive parameters ε and ν. However, it cannot be considered on the whole parameter quadrant. For (ε, ν) belonging to the curve $\nu = \nu_*(\varepsilon)$, which divides the first quadrant into two domains D_1 and D_2, the problem is meaningless because a Green's function does not even exist for these values of ε and ν (see Appendix). For the remaining part of the parameter quadrant (see Figure 2) the question of existence and uniqueness of the solution can be posed. To obtain the answer to this question the methods developed for the usual Neumann–Kelvin problem (see [17, 19, 20]) can be applied. The following theorem summarizes the results obtained in [1] in this respect.

The problem (2.1)–(2.7) *has a unique solution for all* $(\varepsilon, \nu) \in D_1 \cup D_2$, *except possibly on an analytic set of points.*

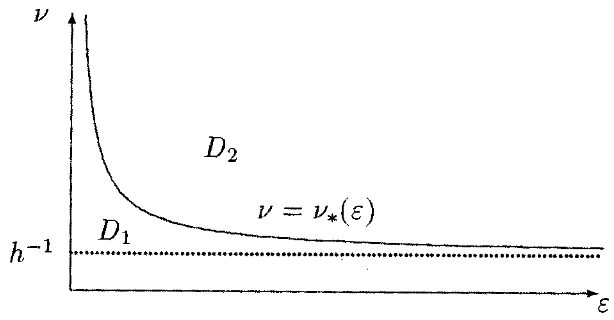

Figure 2. Dependence of the critical value ν_* on the parameter ε.

For the definition of an analytic set see [21, ch. 2, Section 6, p. 46]. This book contains a rather comprehensive theory of analytic sets. Some set of exceptional points for which existence and uniqueness might not be established arises through the method applied for the proof of this theorem (it seems unlikely that such points might really exist). The problem is reduced to an integral equation with an operator depending analytically on two parameters ε and ν. Following [19], we first prove the solvability of this equation for some extreme (large or small) values of ε, and after that we apply the theorems on invertibility of operators depending analytically on one or two parameters (see [22, 23]). According to these theorems invertibility might be violated for some set of pairs of the parameters involved. To give an idea of this exceptional set (see [1] for details), we mention that every line $\nu = \text{const} \neq h^{-1}$ (with possible exception for a finite number of lines) can contain no more than a sequence of exceptional points. When $\nu > h^{-1}$, the only limit point of the sequence is on the curve dividing D_1 and D_2. And if $\nu < h^{-1}$, then such a straight line contains only a finite number of possible exceptional points.

3. Asymptotics at infinity for the velocity potentials

The aim of the present section is to prove the following assertion:

The flow about an arbitrary cylinder depends on the parameters ε and ν in the same way as the flow about source.

It means that in both cases the flows have the same behaviour far downstream for the same values of the parameters. For the proof it is sufficient to compare the far field caused by the body given in (3.18) and (3.19) with the asymptotic formulae (A.5) and (A.9) for the sources. The asymptotic behaviour is the same and the only difference is in the values of the coefficients. This allows us to say that the body has supercritical speed when $(\varepsilon, \nu) \in D_1$, otherwise we shall speak of subcritical forward speed ($(\varepsilon, \nu) \in D_2$). In the former case there exist only waves with the wave number ν, while in the latter case waves with another wave number ν_0 are also present. This number is the only positive root of $Q(\nu_0) = 0$, where

$$Q(k) = (1 + \varepsilon)k + (k - \varepsilon\nu)\tanh kh.$$

Clearly, such a root exists only when $\nu > \nu_*$.

The proof of (3.18) and (3.19) is based on two auxiliary assertions which have an interest of their own. The first of them is concerned with the general form of the velocity potentials for free steady-state two-layer flow. The second deals with integral representations of velocity potentials for flow about the cylinder. These representations yield the asymptotics at infinity for the solution to (2.1)–(2.7) and can also be used to reduce the problem to an integral equation.

Let us consider the velocity potentials $u_0^{(i)}, i = 1,2$ of the free steady-state two-layer flow. They must satisfy (2.2)–(2.4) and

$$\nabla^2 u_0^{(i)} = 0, \quad \text{in} \quad L^{(i)}, \tag{3.1}$$

$$\sup_{L^{(i)}} |\nabla u_0^{(i)}| < \infty \quad i = 1,2. \tag{3.2}$$

Certainly, $u_0^{(i)}(x,y), i = 1,2$ may be treated as distributions in $S'(\mathbb{R}_x)$, *i.e.* as linear continuous functionals depending on the parameter y. Hence, it is possible to apply the Fourier transform (see [24, pp. 108–110]): $\tilde{u}^{(i)} = F[u_0^{(i)}] \in S'(\mathbb{R}_k), i = 1,2$, to (3.1), from which (2.2)–(2.4) can be written as follows:

$$\tilde{u}_{yy}^{(i)} - k^2 \tilde{u}^{(i)} = 0, \quad (0,y) \in L^{(i)}, \quad i = 1,2 \tag{3.3}$$

$$\nu \tilde{u}_y^{(1)} - k^2 \tilde{u}^{(1)} = 0, \quad y = h, \tag{3.4}$$

$$\rho_1 [\nu \tilde{u}_y^{(1)} - k^2 \tilde{u}^{(1)}] = \rho_2 [\nu \tilde{u}_y^{(2)} - k^2 \tilde{u}^{(2)}], \quad y = 0, \tag{3.5}$$

$$\tilde{u}_y^{(1)} = \tilde{u}_y^{(2)}, \quad y = 0. \tag{3.6}$$

In order to solve the equation $u_{yy} - k^2 u = 0$ (we need the general solution which is a distribution in k) we rewrite it as follows:

$$\begin{pmatrix} u \\ v \end{pmatrix}'_y = \begin{pmatrix} 0 & k \\ k & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}, \tag{3.7}$$

and consider u and v as elements in $D'(\mathbb{R}_k) \supset S'(\mathbb{R}_k)$. Then they can be multiplied by infinitely differentiable functions of k. Putting

$$\begin{pmatrix} u \\ v \end{pmatrix} = \exp \left[\begin{pmatrix} 0 & k \\ k & 0 \end{pmatrix} y \right] \begin{pmatrix} p \\ q \end{pmatrix},$$

we obtain two equations: $p'_y = 0$ and $q'_y = 0$ instead of (3.7). Then, $p = p(k)$ and $q = q(k)$. Thus, the general solutions to (3.3) are as follows:

$$\tilde{u}^{(1)} = C_1(k)\, \mathrm{e}^{ky} + C_2(k)\, \mathrm{e}^{-ky}, \qquad \tilde{u}^{(2)} = C_3(k)\, \mathrm{e}^{ky} + C_4(k)\, \mathrm{e}^{-ky},$$

where $C_i(k) \in D'(\mathbb{R}_k), i = 1,2,3,4$. According to (3.2) supp $C_3 \subset \overline{\mathbb{R}_+}$ and supp $C_4 \subset \overline{\mathbb{R}_-}$.

For $k > 0$, (3.4)–(3.6) yield the system

$$\begin{aligned} &(\nu - k)\, \mathrm{e}^{kh} C_1(k) - (\nu + k)\, \mathrm{e}^{-kh} C_2(k) = 0, \\ &C_1(k) - C_2(k) - C_3(k) = 0, \\ &(\nu - k) C_1(k) - (\nu + k) C_2(k) - (1 + \varepsilon)(\nu - k) C_3(k) = 0, \end{aligned} \tag{3.8}$$

which determines $C_1(k), C_2(k)$ and $C_3(k)$. The last two equations imply that

$$C_1(k) = \left[1 + \frac{\varepsilon(k-\nu)}{2k}\right] C_3(k), \qquad C_2(k) = \frac{\varepsilon(k-\nu)}{2k} C_3(k). \tag{3.9}$$

Substituting (3.9) in the first equation (3.8), we arrive at $(\nu - k)Q(k)C_3(k) = 0$. The factor of $C_3(k)$ has simple zeroes at $k = \nu$ and $k = \nu_0$. Hence, for $k > 0$

$$C_3(k) = a_+\delta(k-\nu) + b_+\delta(k-\nu_0),$$

where a_+, b_+ are arbitrary constants. By (3.9) we have for $k > 0$

$$C_1(k) = a_+\delta(k-\nu) + \left[1 + \frac{\varepsilon(\nu_0-\nu)}{2\nu_0}\right] b_+\delta(k-\nu_0),$$
$$C_2(k) = \frac{\varepsilon(\nu_0-\nu)}{2\nu_0} b_+\delta(k-\nu_0).$$

In the same way we get for $k < 0$:

$$C_4(k) = a_-\delta(k-\nu) + b_-\delta(k-\nu_0), \qquad C_1(k) = \frac{\varepsilon(\nu_0-\nu)}{2\nu_0} b_-\delta(k-\nu_0),$$
$$C_2(k) = a_-\delta(k-\nu) + \left[1 + \frac{\varepsilon(\nu_0-\nu)}{2\nu_0}\right] b_-\delta(k-\nu_0)$$

with arbitrary coefficients a_-, b_-.

Lastly, let $k = 0$. By the theorem about distributions with point support (see *e.g.* [24, pp. 44–46]):

$$C_3(k) = \sum_{i=0}^{N} a_i \delta^{(i)}(k), \qquad C_4(k) = \sum_{i=0}^{N} b_i \delta^{(i)}(k).$$

Then,

$$C_3(k)\,\mathrm{e}^{ky} + C_4(k)\,\mathrm{e}^{-ky} = \sum_{j=0}^{N} y^j \sum_{i=j}^{N} \binom{i}{j} \delta^{(i-j)}(k)((-1)^j a_i + b_i)$$

(see [24, p. 37, Example (d)]). According to (3.2) $y^j\delta^{(i)}(k), j \geqslant 2, i \geqslant 0$ and $y\delta^{(l)}, l \geqslant 1$ have zero coefficients. Thus, for $i \geqslant 2$ the equalities $a_i + b_i = 0$ and $a_i - b_i = 0$ hold simultaneously. Hence,

$$C_3(k) = a_0\delta(k) + a_1\delta'(k), \qquad C_4(k) = b_0\delta(k) + b_1\delta'(k). \tag{3.10}$$

By (3.3)–(3.5) we obtain

$$\begin{aligned} & k(\nu-k)\,\mathrm{e}^{kh} C_1(k) - k(\nu+k)\,\mathrm{e}^{-kh} C_2(k) = 0, \\ & k(C_1(k) - C_2(k) - C_3(k) + C_4(k)) = 0, \\ & k(\nu-k)C_1(k) - k(\nu+k)C_2(k) \\ & \quad -(1+\varepsilon)k((\nu-k)C_3(k) - (\nu+k)C_4(k)) = 0. \end{aligned} \tag{3.11}$$

Substituting (3.10) in the last two equations of (3.11), we get

$$k((\nu-k)C_1(k)-(\nu+k)C_2(k))=\nu(1+\varepsilon)(b_1-a_1)\delta(k),$$
$$k(C_1(k)-C_2(k))=(b_1-a_1)\delta(k).$$

From here

$$(k+\nu\varepsilon)kC_1(k)=(\nu\varepsilon-k)kC_2(k). \tag{3.12}$$

On substitution of (3.12) in the first equation (3.11) we arrive at $k^2R(k)C_2(k)=0$, where

$$R(k)=\nu(1+\varepsilon)(\mathrm{e}^{kh}+\mathrm{e}^{-kh})-(k^2+\nu^2\varepsilon)(\mathrm{e}^{kh}-\mathrm{e}^{-kh})/k.$$

Since $\nu\neq\nu_*$, then

$$\lim_{k\to 0}R(k)=2\nu(1+\varepsilon-\nu\varepsilon h)\neq 0.$$

Hence, $C_2(k)=c_0\delta(k)+c_1\delta'(k)$. Furthermore, (3.12) leads to $C_1(k)=d_0\delta(k)+c_1\delta'(k)$. Substituting $C_1(k)$ and $C_2(k)$ in the second equation (3.11), we find that $a_1=b_1$. Now, taking into account (3.2) and

$$(\mathrm{e}^{ky}+\mathrm{e}^{-ky})\delta'(k)=2\delta'(k),$$

we observe that the general solution to (3.3)–(3.6) takes the form:

$$\begin{aligned}\tilde{u}^{(1)} &= d_1\delta(k)+d_2\delta'(k)+\sum_{\pm}\{a_{\pm}\delta(k\mp\nu)\,\mathrm{e}^{\nu h}\\ &\quad +b_{\pm}\delta(k\mp\nu_0)[(1+\varepsilon-\varepsilon\nu\nu_0^{-1})\cosh(\nu_0 y)+\sinh(\nu_0 y)]\},\\ \tilde{u}^{(2)} &= d_3\delta(k)+d_4\delta'(k)+\sum_{\pm}(a_{\pm}\delta(k\mp\nu)\,\mathrm{e}^{\nu h}+b_{\pm}\delta(k\mp\nu_0)\,\mathrm{e}^{\nu_0 h}),\end{aligned}$$

where $d_i, i=1,2,3,4$ and $a_{\pm}, b_{\pm}$ are arbitrary constants, and $\Sigma_{\pm}$ denotes a sum of two terms.

From here it follows that $\tilde{u}^{(1)}$ and $\tilde{u}^{(2)}$ belong to $S'(\mathbb{R}_k)$. Applying the inverse Fourier transform and using the formulae

$$\mathrm{F}^{-1}[\delta(k-a)]=(2\pi)^{-1}\,\mathrm{e}^{iax},\qquad \mathrm{F}^{-1}[\delta'(k)]=(2\pi)^{-1}ix,$$

we arrive at the following assertion.

Let functions $u_0^{(i)}\in C^2(\overline{L^{(i)}}), i=1,2$ *satisfy* (3.1), (3.2) *and* (2.2)–(2.4). *Then*

$$\begin{aligned}u_0^{(1)}(x,y) &= k_1^{(1)}+k_2^{(1)}x+k_1\,\mathrm{e}^{\nu(y+ix)}+k_2\,\mathrm{e}^{\nu(y-ix)}\\ &\quad +H(\nu-\nu_*)E(\nu_0 y)(k_3\,\mathrm{e}^{i\nu_0 x}+k_4\,\mathrm{e}^{-i\nu_0 x}),\\ u_0^{(2)}(x,y) &= k_1^{(2)}+k_2^{(2)}x+k_1\,\mathrm{e}^{\nu(y+ix)}+k_2\,\mathrm{e}^{\nu(y-ix)}\\ &\quad +H(\nu-\nu_*)(k_3\,\mathrm{e}^{\nu_0(y+ix)}+k_4\,\mathrm{e}^{\nu_0(y-ix)}),\end{aligned} \tag{3.13}$$

where $k_1^{(i)}, k_2^{(i)}, k_1, k_2, k_3$ *and* k_4 *are arbitrary complex coefficients and*

$$E(t) = (1 + \varepsilon - \varepsilon\nu\nu_0^{-1}) \cosh t + \sinh t. \tag{3.14}$$

In [17] the integral representation for the velocity potential describing the body's forward motion in a homogeneous fluid was obtained. Here we generalize it to the case of a two-layer fluid. To be specific, we assume that the body is immersed in the upper fluid. Let us extend $u^{(1)}(z)$ into B. By Lemma 6.37 [25, p. 136] this is possible because $u^{(i)} \in C^{2,\alpha}(\overline{W^{(i)}})$ and B is a $C^{2,\alpha}$-domain. Let $u_1^{(1)} \in C^{2,\alpha}(\overline{L^{(1)}})$ be the extension. It is convenient to put $u_1^{(2)} = u^{(2)}$. Then $-\nabla^2 u_1^{(i)} = g$ in $L^{(i)}, i = 1, 2$, and supp $g \subset \overline{B}$. The function

$$u_2^{(i)}(z) = \int_B G^{(i;1)}(z;\zeta) g(\zeta)\, \mathrm{d}\xi\, \mathrm{d}\eta$$

has the same properties as $u_1^{(i)}$. Hence, we can apply (3.13) to the difference $u_1^{(i)} - u_2^{(i)}$. Then, taking into account (2.7), we obtain

$$u^{(i)}(z) = \int_B G^{(i;1)}(z;\zeta) g(\zeta)\, \mathrm{d}\xi\, \mathrm{d}\eta + c_i, \quad z \in W^{(i)}, \tag{3.15}$$

where $c_i, i = 1, 2$ are constants. Clearly,

$$0 = \int_B \nabla^2 G^{(i;1)}(z;\zeta) u_1^{(i)}(\zeta)\, \mathrm{d}\xi\, \mathrm{d}\eta, \quad z \in W^{(i)}. \tag{3.16}$$

Adding (3.15) and (3.16) and using Green's formula, we arrive at the following result:

Let $u^{(i)} \in C^{2,\alpha}(\overline{W^{(i)}}), 0 < \alpha < 1, i = 1, 2$ *be a solution to* (2.1)–(2.7). *Then*

$$u^{(i)}(z) = \int_S \left[u^{(j)}(\zeta) \frac{\partial G^{(i;j)}(z;\zeta)}{\partial n_\zeta} - G^{(i;j)}(z;\zeta) \frac{\partial u^{(j)}(\zeta)}{\partial n_\zeta} \right] \mathrm{d}s_\zeta + c_i, \tag{3.17}$$

where $z \in W^{(i)}, B \subset L^{(j)}$ *and* $c_i, i = 1, 2$ *are constants.*

The last formula and the asymptotic formulae (A.5) and (A.9) for the Green's function (see Appendix) lead to the following theorem:

Let $u^{(i)} \in C^{2,\alpha}(\overline{W^{(i)}}), 0 < \alpha < 1, i = 1, 2$ *be a solution to* (2.1)–(2.7). *Then the following asymptotic formulae hold as* $|z| \to \infty$ *and* $\pm x > 0$*:*

$$\begin{aligned} u^{(1)}(z) \;=\; & \mathcal{Q}^{(1)} \log|z| + H(-x)\{\mathcal{K}x + \mathrm{e}^{\nu y}(\mathcal{A}\sin\nu x + \mathcal{B}\cos\nu x) \\ & + H(\nu - \nu_*) E(\nu_0 y)(\mathcal{A}_0 \sin\nu_0 x + \mathcal{B}_0 \cos\nu_0 x)\} \\ & + C_\pm + \psi^{(1)}(z), \end{aligned} \tag{3.18}$$

$$\begin{aligned} u^{(2)}(z) \;=\; & \mathcal{Q}^{(2)} \log|z| + H(-x)\{\mathrm{e}^{\nu y}(\mathcal{A}\sin\nu x + \mathcal{B}\cos\nu x) \\ & + H(\nu - \nu_*)\, \mathrm{e}^{\nu_0 y}(\mathcal{A}_0 \sin\nu_0 x + \mathcal{B}_0 \cos\nu_0 x)\} \\ & + C_2 + \psi^{(2)}(z), \end{aligned} \tag{3.19}$$

where C_+ and C_2 are arbitrary constants, $E(\nu_0 y)$ is defined by (3.14) *and the estimates $\psi^{(i)} = O(|z|^{-1}), |\nabla\psi^{(i)}| = O(|z|^{-2}), i = 1, 2$ are true.*

If $B \subset L^{(j)}$, then the coefficients in (3.18) *and* (3.19) *are given as follows:*

$$\mathcal{A}_0 = 2\int_S \left[u^{(j)}(x,y)\frac{\partial C_0^{(j)}(y)\cos\nu_0 x}{\partial n} - C_0^{(j)}(y)\cos\nu_0 x\frac{\partial u^{(j)}(x,y)}{\partial n}\right] \mathrm{d}s,$$

$$\mathcal{A} = 2C^{(j)}\int_S \left[u^{(j)}(x,y)\frac{\partial\, \mathrm{e}^{\nu y}\cos\nu x}{\partial n} - \mathrm{e}^{\nu y}\cos\nu x\frac{\partial u^{(j)}(x,y)}{\partial n}\right] \mathrm{d}s,$$

$$\mathcal{Q}^{(i)} = -Q^{(i;j)}\int_S \frac{\partial u^{(j)}}{\partial n}\,\mathrm{d}s, \qquad \mathcal{K} = -\kappa\int_S \frac{\partial u^{(j)}}{\partial n}\,\mathrm{d}s.$$

Here $\kappa = -\varepsilon\nu Q_0^{-1}$ when $j = 1$, and $\kappa = 0$ when $j = 2$; $C^{(j)}, C_0^{(j)}$ and $Q^{(i;j)}$ are given by (A.2), (A.8), (A.6) *and* (A.10). *To obtain $\mathcal{B}, \mathcal{B}_0$, we have to replace* cos *by* − sin *in the formulae for $\mathcal{A}, \mathcal{A}_0$. The equality*

$$C_+ - C_- = k\int_S \left[u^{(j)}\frac{\partial x}{\partial n} - x\frac{\partial u^{(j)}}{\partial n}\right] \mathrm{d}s$$

gives a relationship between C_+ and C_-.

In the next section we shall use the fact that if the mean value of f in (2.5) vanishes, *i.e.* $\int_S f\,\mathrm{d}s = 0$, then $\mathcal{Q}^{(1)} = \mathcal{Q}^{(2)} = \mathcal{K} = 0$.

Let us consider what happens with (3.18) and (3.19) in the limit as $\varepsilon \to 0$. In this case $\nu_* \to \infty$, and hence the limit formulae do not contain terms describing waves with the wave number ν_0. Moreover, in the limit the linear term also cancels and the coefficients of the logarithms take the form given in [17] for the case of a homogeneous fluid. The same is true for $\mathcal{A}$ and $\mathcal{B}$ corrected for a vertical transfer of the origin. Thus, the asymptotic formulae (3.18) and (3.19) have a common limit as $\varepsilon \to 0$, which coincides with the asymptotics of the solution to the Neumann-Kelvin problem in a homogeneous fluid (see [17]).

4. Wave resistance

It is well-known that the wave resistance of a two-dimensional body moving forward in a homogeneous fluid can be expressed in terms of the wave amplitude far downstream or, what is the same, in terms of $\mathcal{A}$ and $\mathcal{B}$ (see *e.g.* [8, Section 26, p. 169], [16, Section 249, p. 415]). In the present section a similar formula is derived for the wave resistance of a cylinder in a two-layer fluid. The coefficients of the wave terms in (3.18) are involved. As usual we assume that (2.5) has the form of the impermeability condition:

$$\partial u^{(i)}(z)/\partial n = U\cos(n,x), \quad z \in S \subset L^{(i)}. \tag{4.1}$$

Then, according to the remark made at the end of Section 3, there are no linear and logarithmic terms in (3.18) and (3.19).

It is easy to show that, as $x \to -\infty$, the free surface elevation $\eta^{(1)}$ and the interface elevation $\eta^{(2)}$ have the following behaviour:

$$\eta^{(1)}(x) \sim U^{-1}[\mathrm{e}^{\nu h}(\mathcal{A}\cos\nu x - \mathcal{B}\sin\nu x)$$
$$-H(\nu-\nu_*)\varepsilon\,\mathrm{e}^{-\nu_0 h}(\mathcal{A}_0\cos\nu_0 x - \mathcal{B}_0\sin\nu_0 x)],$$
$$\eta^{(2)}(x) \sim U^{-1}[\mathcal{A}\cos\nu x - \mathcal{B}\sin\nu x + H(\nu-\nu_*)(\mathcal{A}_0\cos\nu_0 x - \mathcal{B}_0\sin\nu_0 x)].$$

Thus, if the waves corresponding to ν_0 do exist, then on the interface they have the same order of magnitude as the waves corresponding to ν, whereas on the free surface the former waves are much smaller than the latter ones due to different exponential factors.

As in [26] we start from the wave resistance definition:

$$R = -\int_S p \, \cos(n, x) \, \mathrm{d}s,$$

where p is the hydrodynamic pressure along S, which is immersed in the ith layer $(S \subset L^{(i)})$. By Bernoulli's equation for steady flow (see *e.g.* [18])

$$p = \text{const} - \rho_i [V^{(i)}]^2/2.$$

Here ρ_i is the fluid density and $[V^{(i)}]^2 = (u_x^{(i)} - U)^2 + (u_y^{(i)})^2$. By (4.1) we find after simple manipulation that

$$R = \rho_i \int_S [2^{-1} |\nabla u^{(i)}|^2 \cos(n, x) - u_x^{(i)} \partial u^{(i)} / \partial n] \, \mathrm{d}s. \tag{4.2}$$

Let us write Green's identity

$$\begin{aligned} 0 \;&=\; \int_{W_\alpha^{(i)}} u_x^{(i)} \nabla^2 u^{(i)} \, \mathrm{d}x \, \mathrm{d}y \\ &=\; -\int_{W_\alpha^{(i)}} \nabla u^{(i)} \cdot \nabla u_x^{(i)} \, \mathrm{d}x \, \mathrm{d}y - \int_{\partial W_\alpha^{(i)}} u_x^{(i)} \frac{\partial u^{(i)}}{\partial n} \, \mathrm{d}s \end{aligned}$$

for $W_\alpha^{(i)} = W^{(i)} \cap \{|x| < \alpha\}$, where $\alpha > \max\{|x| : z \in S\}$ and $\boldsymbol{n}$ is directed into $W_\alpha^{(i)}$. For $i = 2$ this is possible because there is no logarithmic term in the asymptotics of $u^{(i)}$ as $y \to -\infty$. From the last equation we get by the divergence theorem

$$\begin{aligned} -\int_S u_x^{(i)} \frac{\partial u^{(i)}}{\partial n} \, \mathrm{d}s &= \int_{\partial W_\alpha^{(i)} \setminus S} u_x^{(i)} \frac{\partial u^{(i)}}{\partial n} \, \mathrm{d}s + 2^{-1} \int_{W_\alpha^{(i)}} (|\nabla u^{(i)}|^2)_x \, \mathrm{d}x \, \mathrm{d}y \\ &= \int_{\partial W_\alpha^{(i)} \setminus S} u_x^{(i)} \frac{\partial u^{(i)}}{\partial n} \, \mathrm{d}s - 2^{-1} \int_{\partial W_\alpha^{(i)}} |\nabla u^{(i)}|^2 \cos(n, x) \, \mathrm{d}s. \end{aligned}$$

Thus, (4.2) may be rewritten as follows:

$$R = \sum_{j=1}^{2} \rho_j \int_{\partial W_\alpha^{(j)} \setminus S} \left[u_x^{(j)} \frac{\partial u^{(j)}}{\partial n} - 2^{-1} |\nabla u^{(j)}|^2 \cos(n, x) \right] \mathrm{d}s, \tag{4.3}$$

since the integral for $j \neq i$ obviously vanishes. Let us rewrite the right-hand side in (4.3) as a sum of integrals over segments:

$$\begin{aligned} R \;=\; & -\rho_1 \int_{-\alpha}^{+\alpha} [u_x^{(1)} u_y^{(1)}]_{y=h} \, \mathrm{d}x + 2^{-1} \rho_1 \int_0^h [(u_x^{(1)})^2 - (u_y^{(1)})^2]_{x=-\alpha} \, \mathrm{d}y \\ & + 2^{-1} \rho_1 \int_0^h [(u_y^{(1)})^2 - (u_x^{(1)})^2]_{x=\alpha} \, \mathrm{d}y \end{aligned}$$

$$+2^{-1}\rho_2 \int_{-\infty}^{0} [(u_x^{(2)})^2 - (u_y^{(2)})^2]_{x=-\alpha}\,\mathrm{d}y$$

$$+2^{-1}\rho_2 \int_{-\infty}^{0} [(u_y^{(2)})^2 - (u_x^{(2)})^2]_{x=\alpha}\,\mathrm{d}y$$

$$+ \int_{-\alpha}^{\alpha} [\rho_1 u_x^{(1)} u_y^{(1)} - \rho_2 u_x^{(2)} u_y^{(2)}]_{y=0}\,\mathrm{d}x \tag{4.4}$$

According to (2.2) we have for the first integral here

$$-\rho_1 \int_{-\alpha}^{+\alpha} [u_x^{(1)} u_y^{(1)}]_{y=h}\,\mathrm{d}x$$

$$= \rho_1 \nu^{-1} \int_{-\alpha}^{+\alpha} [u_x^{(1)} u_{xx}^{(1)}]_{y=h}\,\mathrm{d}x = (2\nu)^{-1}\rho_1 \left[(u_x^{(1)})^2\right]_{x=-\alpha}^{x=\alpha}.$$

By (2.3) and (2.4) the last term in (4.4) can be calculated as follows:

$$\int_{-\alpha}^{\alpha} [\rho_1 u_x^{(1)} u_y^{(1)} - \rho_2 u_x^{(2)} u_y^{(2)}]_{y=0}\,\mathrm{d}x$$

$$= \frac{1}{\nu(\rho_2 - \rho_1)} \int_{-\alpha}^{\alpha} (\rho_1 u_x^{(1)} - \rho_2 u_x^{(2)})(\rho_1 u_{xx}^{(1)} - \rho_2 u_{xx}^{(2)})\,\mathrm{d}x$$

$$= \frac{1}{2\nu(\rho_2 - \rho_1)} \left[(\rho_1 u_x^{(1)} - \rho_2 u_x^{(2)})^2\right]_{x=-\alpha}^{x=\alpha}, \quad y = 0.$$

Allowing α to tend to infinity, we arrive at

$$R = -\frac{\nu\rho_1}{4}(\varepsilon + \mathrm{e}^{2\nu h})(\mathcal{A}^2 + \mathcal{B}^2) - H(\nu - \nu_*)\frac{\rho_2 \varepsilon(\nu - \nu_0)}{4}$$

$$\times \left[\frac{\nu}{\nu + \nu_0} - \frac{\varepsilon h(\nu + \nu_0)\,\mathrm{e}^{-2\nu_0 h}}{1 + \varepsilon}\right](\mathcal{A}_0^2 + \mathcal{B}_0^2). \tag{4.5}$$

Here the asymptotic formulae (3.18), (3.19) and the equation $Q(\nu_0) = 0$ are taken into account. The latter relationship allows us to change the square bracket and (4.5) takes the form:

$$R = -\frac{\nu\rho_1}{4}(\varepsilon + \mathrm{e}^{2\nu h})(\mathcal{A}^2 + \mathcal{B}^2) - H(\nu - \nu_*)\frac{\rho_2 \varepsilon(\nu - \nu_0)}{4}$$

$$\times \left[\frac{\nu}{\nu + \nu_0} - h\frac{\varepsilon(\nu - \nu_0) - 2\nu_0}{1 + \varepsilon}\right](\mathcal{A}_0^2 + \mathcal{B}_0^2).$$

It is easy to see, that if $\varepsilon \to 0$, then in the limit R gives the well-known expression for the wave resistance of a cylinder moving forward in a homogeneous fluid (see *e.g.* [8, Section 26, p. 169], [16, Section 249, p. 415]).

A horizontal dipole is a convenient example for the numerical analysis of R. We consider

$$u_d^{(j)}(z) = C G_x^{(j;i)}(z; 0, d), \quad j = 1, 2 \quad \text{where} \quad (0, d) \in L^{(i)}. \tag{4.6}$$

Thus, for $0 < d < h (d < 0)$ the singularity is in the upper (lower) layer.

Our choice is motivated by two well-known facts:

(1) a dipole exactly describes forward motion of a circular cylinder in an unbounded homogeneous fluid (see *e.g.* [16, Section 68, p. 77]);

(2) a dipole submerged under the free surface of such a fluid gives an approximate solution for sufficiently small circular cylinder (see *e.g.* [16, Section 247, pp. 411, 412]). The latter fact is also true for a two-layer fluid. We can easily arrive at this conclusion using the same argument for (4.6) with $C = 2\pi U r^2$. In this case the boundary condition on the circle $x^2 + (y-d)^2 = r^2$ is satisfied up to $O(r^2)$ when $r \ll 1$. On the other hand, we can find some contour close to the above circle where the impermeability condition is satisfied exactly.

For the potential (4.6) the coefficients $\mathcal{Q}^{(1)}, \mathcal{Q}^{(2)}, \mathcal{K}, \mathcal{A}$ and $\mathcal{A}_0$ vanish in (3.18) and (3.19). For a numerical calculation it is convenient to put $C = \rho_1/\rho_i$ in (4.6). Then,

$$\mathcal{B} = -\frac{\nu\, \mathrm{e}^{\nu d}}{\varepsilon + \mathrm{e}^{2\nu d}}, \qquad \mathcal{B}_0 \frac{\nu_0 b(d)}{(\nu - \nu_0) Q'(\nu_0) \cosh \nu_0 h},$$

where

$$b(d) = \begin{cases} \nu \cosh \nu_0 (h-d) - \nu_0 \sinh \nu_0 (h-d), & 0 < d < h, \\ (\nu_0) \cosh \nu_0 h - \nu \sinh \nu_0 h)\, \mathrm{e}^{\nu_0 d}, & d < 0. \end{cases}$$

After a simple manipulation (4.5) takes the form $R = R_s + H(\nu - \nu_*) R_{in}$, where

$$R_s = -\frac{\rho_1 \nu^3\, \mathrm{e}^{2\nu d}}{4(\varepsilon + \mathrm{e}^{2\nu h})} \quad \text{and} \quad R_{in} = -\frac{\rho_1 \varepsilon^2 \nu_0^2\, \mathrm{e}^{-2\nu_0 h} [b(d)]^2}{4(\nu - \nu_0)(\varepsilon\nu - \nu_0) Q'(\nu_0)}.$$

These formulae can be further simplified for $\nu \gg 1$. In this case the equation $Q(\nu_0) = 0$ implies that $\nu_0 \sim \varepsilon\nu/(2+\varepsilon)$. Then,

$$R_s \sim -\frac{\rho_1 \nu^3}{4}\, \mathrm{e}^{2\nu(d-h)}, \qquad R_{in} \sim -\frac{\rho_1 \varepsilon^3 \nu^3}{4(2+\varepsilon)^4} \exp\left[-\frac{2\varepsilon |d| \nu}{2+\varepsilon}\right].$$

Thus, R_s is asymptotically equivalent (for $\nu \gg 1$) to the wave resistance of the horizontal dipole having strength equal to 1, and placed in the homogeneous $(\rho = \rho_1)$ infinite-depth fluid at a depth $h - d$. This means that the contribution to R_s due to the presence of interface (see (4.5)) tends to zero as $\nu \to \infty$.

At the same time R_{in} tends to the wave resistance of a dipole moving in unbounded two-layer fluid. Since the latter resistance depends only on $|d|$, it does not matter whether the dipole is placed above or beneath interface. This property manifests itself in Figure 3, showing the resistance coefficient $C_w = |R|/(\rho_1 g h^2)$ versus the Froude number $\mathrm{Fr} = (\nu h)^{-1/2} (\varepsilon = 0{\cdot}03)$, where curves plotted in the same manner coincide on the left. The upper (lower) curve of each type corresponds to the dipole placed in the upper (lower) layer.

Numerical results for a wider range of the Froude number are shown on the Figures 4 and 5, where C_w is plotted versus the Froude number for $\varepsilon = 0{\cdot}03$ and different positions of the dipole. The arrow marks the critical value of the Froude number

$$\mathrm{Fr}_* = (\nu_* h)^{-1/2} = \left(\frac{\varepsilon}{1+\varepsilon}\right)^{1/2} \approx 0{\cdot}171.$$

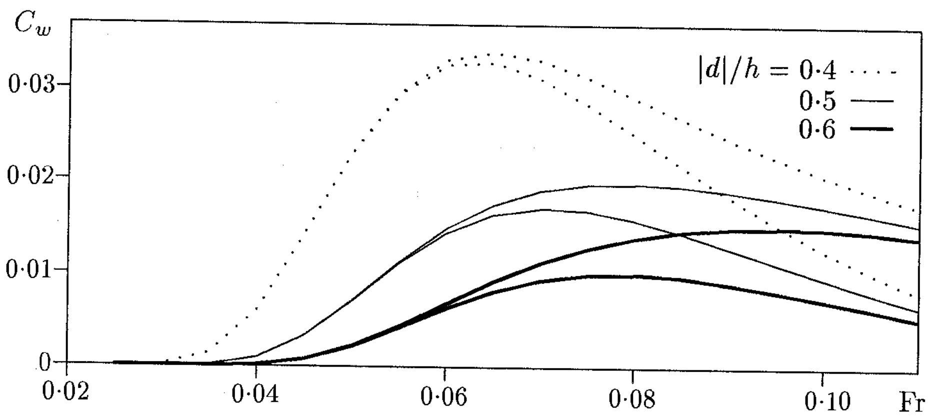

Figure 3. Dependence of the wave resistance coefficient C_w on the Froude number in the subcritical regime.

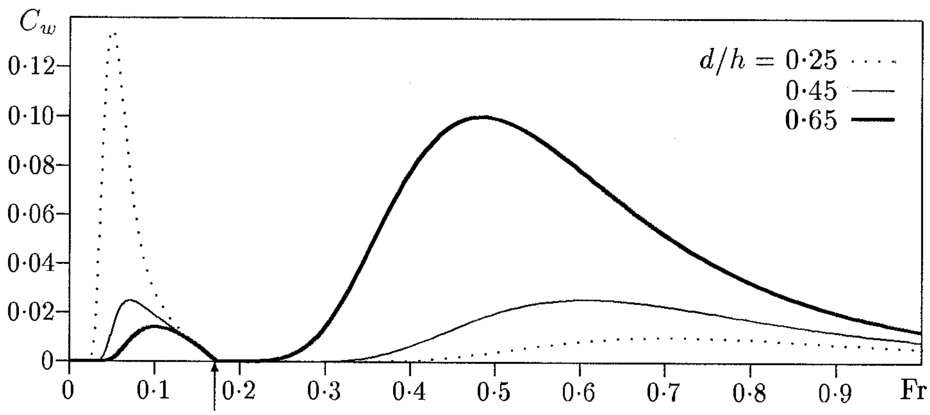

Figure 4. Dependence of C_w on the Froude number for different dipole's positions in the upper fluid.

Internal waves exist only for $\mathrm{Fr} < \mathrm{Fr}_*$. The curves on Figure 4 (Figure 5) correspond to the position of the dipole between the free surface and interface (below the interface).

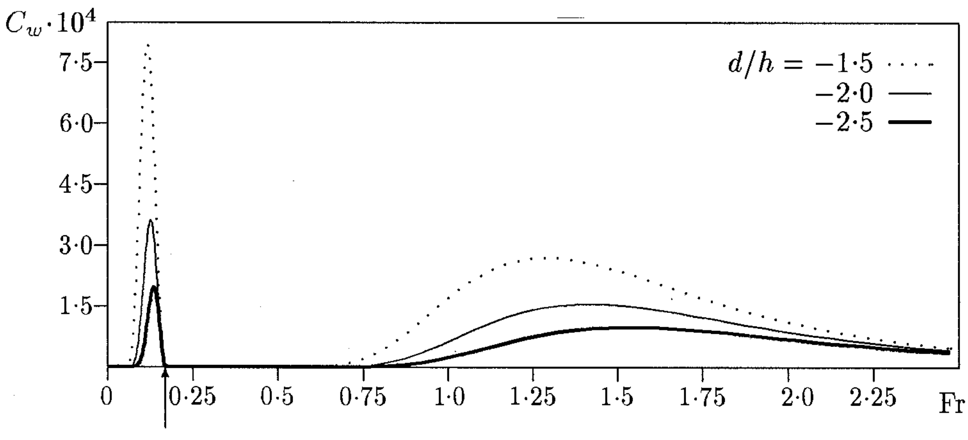

Figure 5. Dependence of C_w on the Froude number for different dipole's positions in the lower fluid.

When the dipole is placed close to the middle of the upper layer, C_w has two almost equal maxima. The first (second) of them is due to a contribution of internal (surface) waves into R, and is decreasing (increasing) as $d > 0$ increases.

The results shown on Figure 5 for the position of the dipole below the interface are in good qualitative agreement with those obtained by Sturova [13, Figure 5] for an elliptic cylinder. It is natural that the first maximum corresponding to a contribution of the internal waves is larger than the second one which is due to surface waves. Both of them decrease as the depth of submergence increases.

The numerical results presented in Figures 4, 5 confirm the conclusion made in [11, 12] (where 3-D problems were treated) that dead-water effects do occur in a two-layer fluid. In our case they manifest themselves in the form of an additional maximum on the wave-resistance graph in the subcritical range of the Froude number.

In [16, Section 249, p. 415] another method is applied for the calculation of the wave resistance of a cylinder in a homogeneous fluid. It is based on the energy consideration, and the starting point is the formula:

$$R = -\frac{U-c}{U}\bar{E},$$

where c stands for group velocity of waves at downstream infinity and $\bar{E}$ is the average energy of waves per unit area of the free surface. Our formula (4.5) can be written in a similar form. Let

$$v_0^{(1)} = \mathrm{e}^{\nu_0 y}(\mathcal{A}_0 \sin \nu_0 x + \mathcal{B}_0 \cos \nu_0 x),$$
$$v_0^{(2)} = \mathrm{e}^{\nu_0 y}(\mathcal{A}_0 \sin \nu_0 x + \mathcal{B}_0 \cos \nu_0 x),$$
$$v = \mathrm{e}^{\nu y}(\mathcal{A} \sin \nu x + \mathcal{B} \cos \nu x)$$

be the wave terms in (3.18) and (3.19). We denote by $\omega_0 = \nu_0 U, \omega = \nu U$ and c_0, c encounter frequencies and group velocities corresponding to ν_0 and ν, respectively; let $\bar{E}_0$ and $\bar{E}$ denote the wave energy per unit area corresponding to the same wave numbers.

Since $U = \sqrt{g/\nu}$, then $c = \mathrm{d}\omega/\mathrm{d}\nu = U/2$. Furthermore, $\omega_0 = \nu_0\sqrt{g/\nu(\nu_0)}$, where $\nu(\nu_0)$ is the implicit function given by $Q(\nu_0) = 0$. Then,

$$c_0 = \frac{\mathrm{d}\omega_0}{\mathrm{d}\nu_0} = U\left(1 - \frac{\nu_0 \nu'(\nu_0)}{2\nu}\right)$$

$$\text{and} \quad \frac{U-c_0}{U} = \frac{1}{2}\left[1 - \frac{\varepsilon h(\nu+\nu_0)^2\,\mathrm{e}^{-2\nu_0 h}}{\nu(1+\varepsilon)}\right].$$

Obviously, we have

$$\bar{E} = \frac{\nu}{2\pi}\int_0^{2\pi/\nu} \mathrm{d}x \left(\rho_1 \int_0^h |\nabla v|^2\,\mathrm{d}y + \rho_2 \int_{-\infty}^0 |\nabla v|^2\,\mathrm{d}y\right),$$

$$\bar{E}_0 + \frac{\nu_0}{2\pi}\int_0^{2\pi/\nu_0} \mathrm{d}x \left(\rho_1 \int_0^h |\nabla v_0^{(1)}|^2\,\mathrm{d}y + \rho_2 \int_{-\infty}^0 |\nabla v_0^{(2)}|^2\,\mathrm{d}y\right).$$

Hence, taking into account the definition of ν_0, we get

$$\bar{E} = \frac{\nu\rho_1}{2}(\varepsilon + \mathrm{e}^{2\nu h})(\mathcal{A}^2 + \mathcal{B}^2), \qquad \bar{E}_0 = \frac{\nu\rho_2\varepsilon(\nu-\nu_0)}{2(\nu+\nu_0)}(\mathcal{A}_0^2 + \mathcal{B}_0^2).$$

Thus, we arrive at the following assertion:

The formula

$$R = -\frac{U-c}{U}\bar{E} - H(\nu - \nu_*)\frac{U-c_0}{U}\bar{E}_0. \tag{4.7}$$

is equivalent to (4.5).

In the present section we used the definition of the wave resistance as the starting point for the derivation of (4.5). However, from the last assertion we see that (4.7) could also be used for this purpose.

5. Summary and conclusions

In this paper we have studied the linearized two-dimensional boundary-value problem which describes the forward motion of a cylinder immersed in one of the layers of a two-layer fluid. First of all, the exact solution to the problem in the absence of a body is derived by means of the Fourier transform. After that any pair of functions satisfying Laplace's equation, the boundary condition on the free surface, the coupling conditions on the interface and the conditions at infinity are represented by Green's formula. Substitution of the asymptotics of the source potentials in this integral representation gives asymptotic formulae for the solution of the boundary-value problem as well as convenient expressions for the coefficients in the asymptotics. Thus, it is shown that the flow about an arbitrary cylinder has the same structure as the flow about a source, *i.e.* there are two regimes of flow: with and without internal waves. The latter regime takes place when the forward velocity exceeds some critical value depending on the densities of the superposed fluids and depth of the upper layer. Surface waves exist in both cases because the lower fluid is assumed to have infinite depth.

It should be pointed out that Green's integral representation allows us to reduce the problem to a boundary-integral equation. The latter, in its turn, allows us to prove the existence of the solution and to find it numerically. However, for the sake of brevity only the formulation of the existence theorem is given.

The asymptotic formulae are applied to finding the wave resistance in terms of the wave amplitudes at downstream infinity. Another expression for the wave resistance in terms of energy losses is also given. To illustrate the obtained results, the wave resistance of a dipole placed above/beneath the interface is investigated both asymptotically and numerically. This example sheds some light on the so-called dead-water phenomenon. It manifests itself in the form of an additional maximum of the wave-resistance coefficient. This maximum is located in the range of subcritical values of the Froude number. The wave resistance coefficient also demonstrates natural dependence on the body's depth of submergence.

There are two problems for further research which are closely related to that considered here. First, the formula (4.5) for the wave resistance can easily be extended to the case when both fluids have finite depth. Three regimes are expected to occur in this situation:

(i) supercritical flow without waves at downstream infinity (a similar regime does exist for a homogeneous fluid of finite depth when the wave resistance vanishes);
(ii) intermediate regime when waves are present only on the free surface of the upper fluid;
(iii) the low-speed regime with waves on the interface as well on the free surface. The last two possibilities are analogous to those in the present paper.

Secondly, the problem (2.1)–(2.7) for a contour intersecting the interface is of great interest. For such a geometry the problem probably requires supplementary conditions as in the case of a surface-piercing body in the homogeneous fluid. The approach developed for the latter problem in [20] seems to be useful for the two-layer fluid as well.

Appendix. Green's function

The Green function (the source function) of the Neumann-Kelvin problem is the velocity potential describing the fluid motion due to a point source at (ξ, η). For this function the following notation $G^{(i;j)}(x, y; \xi, \eta)$ is convenient. Here i stands for the layer (1–upper, 2–lower) containing an observation point (x, y), and j stands for the layer containing the source, *i.e.* $(x, y) \in \overline{L^{(i)}}$ and $(\xi, \eta) \in L^{(j)}$. Thus, Green's function is the pair $\{G^{(1;j)}, G^{(2;j)}\}$ with elements representing this function in the first and the second layer, respectively.

The Green function must satisfy the following boundary-value problem:

$$
\begin{aligned}
&\nabla^2_{x,y} G^{(i;j)}(x,y;\xi,\eta) = -\delta(x-\xi, y-\eta) \quad \text{in } L^{(i)}, \quad i = 1, 2,\\
&G^{(1;j)}_{xx} + \nu G^{(1;j)}_{y} = 0 \quad \text{on } F^{(1)},\\
&G^{(1;j)}_{y} = G^{(2;j)}_{y}, \qquad \rho_1[G^{(1;j)}_{xx} + \nu G^{(1;j)}_{y}] = \rho_2[G^{(2;j)}_{xx} + \nu G^{(2;j)}_{y}] \quad \text{on } F^{(2)},\\
&\limsup_{(x,y)\to\infty} |\nabla_{x,y} G^{(i;j)}| < \infty, \qquad \lim_{x\to+\infty} |\nabla_{x,y} G^{(i;j)}| = 0, \quad i = 1, 2.
\end{aligned}
$$

We can apply the Fourier transform for the derivation of the explicit solution to this problem (see *e.g.* [18]). However, the straightforward calculations are rather long and tiresome (see the calculation of the velocity potentials for the free stream in Section 3). If the source is in the upper layer, then we arrive at the following result:

$$
\begin{aligned}
G^{(1;1)}(z;\zeta) &= G^{(1;1)}_0(z;\zeta) - \varepsilon\nu(2Q_0)^{-1}(x-\xi) + C^{(1)}\,\mathrm{e}^{\nu(y+\eta)} \sin\nu(x-\xi)\\
&\quad + H(\nu-\nu_*) C^{(1)}_0(\eta) E(\nu_0 y) \sin\nu_0(x-\xi),\\
G^{(2;1)}(z;\zeta) &= G^{(2;1)}_0(z;\zeta) + C^{(1)}\,\mathrm{e}^{\nu(y+\eta)} \sin\nu(x-\xi)\\
&\quad + H(\nu-\nu_*) C^{(1)}_0(\eta)\,\mathrm{e}^{\nu_0 y} \sin\nu_0(x-\xi).
\end{aligned}
\tag{A.1}
$$

Here $z = x + iy, \zeta = \xi + i\eta, Q_0 = \lim_{k\to 0} Q(k)/k = 1 + \varepsilon - \varepsilon\nu h, E(\nu_0 y)$ is defined by (3.14) and

$$
\begin{aligned}
C^{(1)} &= -(\mathrm{e}^{2\nu h} + \varepsilon)^{-1},\\
C^{(1)}_0(\eta) &= \frac{\nu\cosh(\nu_0(\eta-h)) + \nu_0\sinh(\nu_0(\eta-h))}{(\nu-\nu_0)Q'(\nu_0)\cosh(\nu_0 h)},
\end{aligned}
\tag{A.2}
$$

where

$$
Q'(\nu_0) = \left.\frac{\mathrm{d}Q}{\mathrm{d}k}\right|_{k=\nu_0} = \frac{\varepsilon}{\varepsilon\nu - \nu_0}[\nu(1+\varepsilon) - \varepsilon h(\nu+\nu_0)^2\,\mathrm{e}^{-2\nu_0 h}].
$$

The functions $G_0^{(1;1)}$ and $G_0^{(2;1)}$ are as follows:

$$
\begin{aligned}
G_0^{(1;1)}(z;\zeta) &= -(2\pi)^{-1} \\
&\quad \times \left\{ \log|z-\zeta| - \fint_0^{+\infty} \sum_{i=1}^{4} E^{(i)}(k)\,\mathrm{e}^{kY_i}\cos k(x-\xi)\,\mathrm{d}k \right\}, \\
G_0^{(2;1)}(z;\zeta) &= (2\pi)^{-1} \fint_0^{+\infty} \left[\frac{\nu+k}{\nu-k}\,\mathrm{e}^{k(y+\eta-h)} + \mathrm{e}^{k(y-\eta+h)} \right] \\
&\quad \times \frac{\cos k(x-\xi)\,\mathrm{d}k}{Q(k)\cosh(kh)},
\end{aligned}
\tag{A.3}
$$

where $\boldsymbol{Y} = (y+\eta-2h, \eta-y-h, y-\eta-h, -y-\eta)^t$, $\fint$ means that integral is regularized (the regularization procedure is described below) and

$$
\begin{aligned}
E^{(1)} &= \frac{(\nu+k)\,\mathrm{e}^{kh}}{Q(k)\cosh(kh)} \left[\frac{1}{\nu-k} - \frac{\varepsilon}{2k} \right], \\
E^{(2)} &= E^{(3)} = -\frac{\varepsilon(\nu+k)}{2kQ(k)\cosh(kh)}, \\
E^{(4)} &= \frac{\varepsilon(k-\nu)\,\mathrm{e}^{kh}}{2kQ(k)\cosh(kh)}.
\end{aligned}
\tag{A.4}
$$

These functions have singularities of the form:

$$(k-\nu)^{-1}, k^{-1}, k^{-2} \quad \text{and} \quad (k-\nu_0)^{-1}$$

(the latter exists if $\nu > \nu_*$). If $\nu = \nu_*$, then $E^{(i)}, i = 1,2,3,4$ has a third-order singularity at $k = 0$. Hence, the above formulae for Green's function are not valid in this case, and this critical value should be excluded from consideration.

Let us describe the regularization procedure for the integrals in (A.3). If the integrand has a simple pole at $a > 0$, then the integral is understood in the sense of the Cauchy principal value. When the denominator has a zero of the first or second order at $k = 0$, then the following regularizations are applied:

$$
\begin{aligned}
\fint_0^{+\infty} \frac{f(k)}{k}\,\mathrm{d}k &= \int_0^{+\infty} \frac{f(k) - f(0)\,\mathrm{e}^{-k}}{k}\,\mathrm{d}k, \\
\fint_0^{+\infty} \frac{f(k)}{k^2}\,\mathrm{d}k &= \int_0^{+\infty} \frac{f(k) - f(0) - k\,\mathrm{e}^{-k} f'(0)}{k^2}\,\mathrm{d}k.
\end{aligned}
$$

The pair (A.1) is unique (up to additive constants). This is a consequence of the proposition on the velocity potentials of the free stream proved in Section 3.

For obtaining the asymptotic formulae for (A.1) as $|z| \to \infty$ it is necessary for us to know the asymptotic behaviour of the integrals in (A.3). Here we restrict ourselves to a brief description of the method applied for asymptotic evaluation. It is sufficient to take the first integral as an example. Let us write each function $E^{(i)}, i = 1,2,3,4$ (see (A.4)) as a sum of two functions. One of these contains all singular terms in the Laurent expansions at $0, \nu$ and ν_0, and the other is analytic in k on $(-\delta, +\infty), \delta > 0$. The contribution to the asymptotics of $G_0^{(1;1)}$ when the latter function is integrated is $O(|z|^{-1})$ and the derivatives of this integral are $O(|z|^{-2})$. Each integral containing a singularity has a well-known expression in terms of transcendental functions that have explicit asymptotic expansions. This method leads to the following proposition.

Let $\zeta \in L^{(1)}$. Then as $|\zeta| \leqslant c < \infty$ and $|z| \to \infty$, the following asymptotic formulae:

$$\begin{aligned} G^{(1;1)}(z;\zeta) &= Q^{(1;1)} \log|z| + \varepsilon\nu h(\pi Q_0)^{-1} + 2H(-x) \\ &\quad \times\{\varepsilon\nu(2Q_0)^{-1}(\xi - x) + C^{(1)}\,\mathrm{e}^{\nu(y+n)} \sin\nu(x-\xi) \\ &\qquad + H(\nu-\nu_*)C_0^{(1)}(\eta)E(\nu_0 y)\sin\nu_0(x-\xi)\} + \psi_\pm^{(1;1)}, \\ G^{(2;1)}(z;\zeta) &= Q^{(2;1)}\log|z| + 2H(-x) \\ &\quad \times\{C^{(1)}\,\mathrm{e}^{\nu(y+\eta)}\sin\nu(x-\xi) + H(\nu-\nu_*) \\ &\qquad \times C_0^{(1)}(\eta)\,\mathrm{e}^{\nu_0 y}\sin\nu_0(x-\xi)\} + \psi_\pm^{(2;1)} \end{aligned} \tag{A.5}$$

are true. Here $\psi_\pm^{(i;1)} = O(|z|^{-1}), |\nabla\psi_\pm^{(i;1)}| = O(|z|^{-2}), i = 1, 2$, and

$$Q^{(1;1)} = -(1+\varepsilon)(\pi Q_0^2)^{-1}, \qquad Q^{(2;1)} = -(\pi Q_0)^{-1}. \tag{A.6}$$

The constants $C^{(1)}$ and $C_0^{(1)}$ are given in (A.2).

The above results concerning (A.1) can be literally (and even with some simplifications) reformulated for a source in the lower layer. The corresponding pair of functions is unique (up to arbitrary terms) and has the form:

$$\begin{aligned} G^{(1;2)}(z;\zeta) &= G_0^{(1;2)}(z;\zeta) + C^{(2)}\,\mathrm{e}^{\nu(y+\eta)}\sin\nu(x-\xi) \\ &\quad + H(\nu-\nu_*)C_0^{(2)}(\eta)E(\nu_0 y)\sin\nu_0(x-\xi), \\ G^{(2;2)}(z;\zeta) &= G_0^{(2;2)}(z;\zeta) + C^{(2)}\,\mathrm{e}^{\nu(y+\eta)}\sin\nu(x-\xi) \\ &\quad + H(\nu-\nu_*)C_0^{(2)}(\eta)\,\mathrm{e}^{\nu_0 y}\sin\nu_0(x-\xi). \end{aligned} \tag{A.7}$$

Here

$$C^{(2)} = -\frac{1+\varepsilon}{\mathrm{e}^{2\nu h} + \varepsilon}, \qquad C_0^{(2)}(\eta) = \frac{(1+\varepsilon)(\nu_0 - \nu\tanh(\nu_0 h))}{(\nu-\nu_0)Q'(\nu_0)}\,\mathrm{e}^{\nu_0\eta}, \tag{A.8}$$

and

$$\begin{aligned} G_0^{(1;2)}(z;\zeta) &= \frac{1+\varepsilon}{2\pi} \fint_0^{+\infty} \left[\frac{\nu+k}{\nu-k}\,\mathrm{e}^{k(y+\eta-h)} + \mathrm{e}^{k(\eta-y+h)}\right] \frac{\cos k(x-\xi)\,\mathrm{d}k}{Q(k)\cosh(kh)}, \\ G_0^{(2;2)}(z;\zeta) &= -(2\pi)^{-1}\Big\{ \log|z-\zeta| + \log|z-\overline{\zeta}| \\ &\qquad + 2(1+\varepsilon) \fint_0^{+\infty} \frac{(k-\nu\tanh(kh))}{(k-\nu)Q(k)}\,\mathrm{e}^{k(y+\eta)}\cos k(x-\xi)\,\mathrm{d}k \Big\}. \end{aligned}$$

As above we formulate an assertion giving the asymptotic behaviour of Green's function at infinity.

Let $\zeta \in L^{(2)}$. *Then as* $|\zeta| \leqslant c < \infty$ *and* $|z| \to \infty$, *the following asymptotic formulae:*

$$\begin{aligned} G^{(1;2)}(z;\zeta) &= Q^{(1;2)} \log|z| + 2H(-x)\{C^{(2)} \,\mathrm{e}^{\nu(y+\eta)} \sin\nu(x-\xi) \\ &\quad + H(\nu-\nu_*)C_0^{(2)}(\eta)E(\nu_0 y)\sin\nu_0(x-\xi)\} + \psi_\pm^{(1;2)}, \\ G^{(2;2)}(z;\zeta) &= Q^{(2;2)} \log|z| + 2H(-x)\{C^{(2)} \,\mathrm{e}^{\nu(y+\eta)} \sin\nu(x-\xi) \\ &\quad + H(\nu-\nu_*)C_0^{(2)}(\eta)\,\mathrm{e}^{\nu_0 y}\sin\nu_0(x-\xi)\} + \psi_\pm^{(2;2)} \end{aligned} \tag{A.9}$$

are true. Here $\psi_\pm^{(i;2)} = O(|z|^{-1}), |\nabla\psi_\pm^{(i;2)}| = O(|z|^{-2}), i = 1, 2,$ *and*

$$Q^{(1;2)} = -(1+\varepsilon)(\pi Q_0)^{-1}, \qquad Q^{(2;2)} = -\pi^{-1}. \tag{A.10}$$

The constants $C^{(2)}$ *and* $C_0^{(2)}$ *are given in* (A.8).

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References

1. O.V. Motygin, *Waves effects and the wave-making resistance for 2D bodies moving in homogeneous and two-layer fluid.* PhD Dissertation, St Petersburg University (1996) 114 pp. (in Russian).
2. V.W. Ekman, On dead water. In: *Scientific Results of the Norwegian North Polar Expedition* 5. Christiania (1904) pp. 1–152.
3. H. Lamb, On waves due to a travelling disturbance with an application to waves superposed fluids. *Philosophical Magazine* (6) 31 (1916) 386–399.
4. L.N. Sretenskiy, Waves on the interface between two superposed fluids with application to the “dead-water” phenomenon. *Journal of Geophysics* 4 (1934) 332–370 (in Russian).
5. L.N. Sretenskiy, On the wave resistance in the presence of internal waves. *Izvestia AN SSSR, OTN, Mechanics and Mechanical Engineering* 1 (1959) 56–63 (in Russian).
6. P.N. Uspenskiy, On the wave resistance of a ship in the presence of internal waves (the case of finite depth). *Proc. of Inst. of Marine Hydrophysics* 18 (1959) 68–85 (in Russian).
7. A.A. Hudimac, Ship waves in a stratified ocean. *J. of Fluid Mech.* 11 (1961) 229–243.
8. A.A. Kostyukov, *Theory of Ship Waves and Wave Resistance.* Leningrad: Sudostroyeniye (1959) 312 pp. (in Russian, English translation: Iowa City: Effective Communications Inc. (1968)).
9. V.A. Gorodtsov and E.V. Teodorovich, On the theory of wave resistance (surface and internal waves). In: A. Yu. Ishlinskiy (ed.) *N.E. Kochin and Progress in Mechanics.* Moscow: Nauka (1984) pp. 131–149 (in Russian).
10. B.C. Barber, On the dispersion relation for trapped internal waves. *J. of Fluid Mech.* 252 (1993) 31–49.
11. T. Miloh, M.P. Tulin and G. Zilman, Dead-water effects of a ship moving in a stratified sea. *J. of Offshore Mech. Arctic Eng.* 115 (1993) 105–110.
12. G. Zilman and T. Miloh, Hydrodynamics of a body moving over a mud layer – Part I: Wave resistance. *J. of Ship Res.* 39 (1995) 194–201.
13. I.V. Sturova, The effect of internal waves on the hydrodynamic characteristics of a submerged body, *Izvestia RAN, Phys. of Atmos. and Ocean* 29 (1993) 732–738 (in Russian).
14. V.S. Voitsenya, The two-dimensional problem of the forward body’s motion below the interface between two fluids. *Proc. of Novocherkassk Polytech. Inst.* 104 (1959) 95–111 (in Russian).
15. V.S. Voitsenya, On the forward motion of a body above the interface between two fluids. *Izvestia VUZ’ov, Math.* 2 (1963) 262–270 (in Russian).
16. H. Lamb, *Hydrodynamics*. Cambridge: Cambridge University Press (1932) 738 pp.
17. B.R. Vainberg and V.G. Maz’ya, On the plane problem of the motion of a body immersed in a fluid. *Trans. of Moscow Math. Soc.* 28 (1973) 35–56.
18. J.V. Wehausen and E.V. Laitone, Surface waves. In: S. Flügge (ed.), *Handbuch der Physik* 9. Berlin: Springer-Verlag (1960) pp. 446–778.

19. N.E. Kochin, On the wave resistance and lift of bodies submerged in a fluid. In: *Trudy Konferentsii po Teorii Volnovogo Soprotivleniya*. Moscow: TsAGI (1937) pp. 65–134 (in Russian, English translation in *SNAME Tech. and Res. Bull.* 1–8 (1951)).
20. N.G. Kuznetsov and V.G. Maz'ya, On unique solvability of the plane Neumann–Kelvin problem. *Math. USSR, Sbornik* 63 (1989) 425–446.
21. M. Herve, *Several Complex Variables: Local Theory*. Oxford: Oxford University Press (1963) 196 pp.
22. V.P. Trofimov, The root subspaces of operators depending analytically on a parameter. *Matematicheskie Issledovaniya* 3 No. 3(9) (1968) 117–125 (in Russian).
23. S.G. Krein and V.P. Trofimov, On holomorphic operator-functions of several complex variables. *Functional Analysis and its Applications* 3 No. 4 (1969) 85–86 (in Russian).
24. V.S. Vladimirov, *Generalized Functions in Mathematical Physics.* Moscow: Mir Publishers (1979) 362 pp.
25. D. Gilbarg and N.S. Trudinger, *Elliptic Partial Differential Equations of Second Order.* Berlin: Springer (1983) 514 pp.
26. N.G. Kuznetsov, Wave resistance of a semisubmerged cylinder on deep water. In: A.G. Terent'yev (ed.), *High Speed Hydrodynamics.* Cheboksary: Chuvashian University Press (1990) pp. 53–60 (in Russian).